CWI Quarterly Volume 10 (3&4) 1997, pp. 207 – 213

The Necessity of Fitted Operators and Shishkin Meshes for Resolving Thin Layer Phenomena

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1. Introduction

In this paper we consider two distinct classes of problems on the same domain $D = \Omega \times (0, T]$ where the open interval $\Omega = (0, 1)$. Both classes of problems involve simple parabolic partial differential equations whose solutions have thin layers. It turns out that for the first class of problems there is no fitted operator method on a uniform rectangular mesh that gives satisfactory numerical solutions, but it is easy to construct a simple piecewise uniform Shishkin mesh and a standard finite difference operator, which give a numerical method having satisfactory numerical solutions. Indeed an equally simple Shishkin mesh can be constructed, which gives essentially second order convergence in the space variable.

For the second class of problems no rectangular mesh can be found on which a standard finite difference operator gives a numerical method having satisfactory solutions. This implies, in particular, that there is no fitted mesh method using a rectangular piecewise uniform fitted mesh and a standard finite difference operator which give a numerical method with satisfactory approximate solutions.

In short we can say that these two simple classes of problems demonstrate the following: if we have to find satisfactory numerical solutions to problems involving thin layers, then we cannot use standard finite difference operators on uniform rectangular meshes. In some cases it suffices to use standard finite difference operators on piecewise-uniform Shishkin meshes, but in general it is necessary to use fitted finite difference operators on piecewise-uniform Shishkin meshes.

First, we define more precisely what we mean by a satisfactory numerical method. This is based on what is regarded as satisfactory by end-users of a numerical method, rather than various weaker definitions currently adopted by many professional numerical analysts. The former definition is based on what is required by, for example, design engineers who may need reliable numerical solutions in safety-critical situations, while the latter definitions are often based on what is currently possible to establish by rigorous mathematical analysis.

We take the view that the least an end-user should expect of a numerical method, which is designed to solve problems with thin layer phenomena, is the following: the numerical solutions should be parameter-robust and globally pointwise-convergent. By parameter-robust we mean that the convergence behaviour is robust with respect to the parameters determining the thinness of the layer phenomena (for example, the singular perturbation parameters), and by globally pointwise-convergent we mean that convergence is measured in the maximum norm and that convergence occurs at every point of the domain, not just at the mesh points.

Expressing the above definition in more mathematical terms we say that a numerical method is parameter-robust (or ε-uniform) if there exist positive constants N_0, C and p, independent of ε and N, such that for all $N \geq N_0$

$$\sup_{0<\varepsilon\leq 1} ||\bar{U}_\varepsilon - u_\varepsilon|| \leq CN^{-\rho} ,$$

where $||\cdot||$ indicates the maximum norm on $\bar{D}$, u_ε is the exact solution on D and $\bar{U}_\varepsilon$ is the piecewise bilinear interpolant on D of a finite difference or finite element solution generated by the numerical method on a mesh D_ε^N. Engineers generally also, quite correctly, insist that the appropriately normalized finite difference quotients of the approximate solution and the analogously normalized derivatives of the exact solution satisfy a similar ε-uniform error estimate. They also require that dependent variables like mass, which are obviously non-negative, should be approximated by non-negative numerical solutions; in other words that the numerical solutions are of physical relevance. The latter requirement can normally be achieved in practice by using a monotone numerical method. In this paper we do not discuss the former requirement, beyond stating that the ε-uniform methods described here, based on fitted finite difference operators and piecewise-uniform Shishkin meshes do have these additional properties.

2. Two simple classes of problems

The first class of problems is the following initial-boundary value problem for the singularly perturbed heat equation

$$(P_1) \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \varepsilon \dfrac{\partial^2 u_\varepsilon}{\partial x^2} & \text{in} \quad D \; ; \\[2ex] u_\varepsilon = \varphi & \text{on} \quad \Gamma' \; , \end{cases}$$

where $\Gamma' = \Gamma_l \cup \Gamma_r \cup \Gamma_b$ with Γ_b the bottom edge of D and Γ_l, Γ_r the left, right edges respectively. We take homogenous initial conditions $\varphi|_{\Gamma_b} = 0$ and we assume φ is smooth and that the following compatibility conditions hold

$$\varphi|_{\Gamma_l}(0,0) = \varphi|_{\Gamma_r}(1,0) = 0$$

The solution of this problem has a parabolic boundary layer of width $0(\sqrt{\varepsilon})$ on each edge Γ_l and Γ_r, since the corresponding reduced problem has the trivial solution zero.

The second class of problems is also an initial-boundary value problem for a singularly perturbed parabolic differential equation

$$(P_2) \qquad \begin{cases} x\dfrac{\partial u_\varepsilon}{\partial t} = \varepsilon\dfrac{\partial^2 u_\varepsilon}{\partial x^2} - u_\varepsilon & \text{in} \quad D\ ; \\ u_\varepsilon = \varphi & \text{on} \quad \Gamma'\ . \end{cases}$$

We assume that φ is smooth and satisfies sufficient compatibility conditions at the corners $(0,0),(1,0)$. The solution of this problem has in general a parabolic boundary layer of width $0(\sqrt{\varepsilon})$ on each edge Γ_l and Γ_r, and also a boundary layer of width $0(x)$ on Γ_b. This follows from the fact that the solution of the corresponding reduced problem is $e^{-\frac{t}{x}}$. Because Γ_b is the boundary corresponding to $t=0$ this boundary layer is sometimes called an initial layer.

3. The numerical methods

For the first class of problems (P_1) we consider two numerical methods, one of which is ε-uniform the other is not. The first is a fitted operator method on a uniform rectangular mesh $\bar{D}_u^N = \{x_i\}_0^{N_x} \times \{t_j\}_0^{N_t}$ where $x_j = \frac{i}{N_x}, \quad t_j = \frac{j}{N_t}T$ and $N = (N_x, N_t)$. It uses the second order centred finite difference operator in x

$$\delta_x^2\psi(x_i) = \frac{2(D_x^+ - D_x^-)}{x_{i+1} - x_{i-1}}\psi(x_i)$$

and the first order backward difference operator in t

$$D_t^-\psi(t_j) = \frac{\psi(t_j) - \psi(t_{j-1})}{t_j - t_{j-1}}\ .$$

The resulting numerical method is the fitted finite difference method

$$(P_1^N) \qquad \begin{cases} D_t^- U_\varepsilon = \varepsilon\gamma\delta_x^2 U_\varepsilon & \text{on} \quad D_u^N\ ; \\ U_\varepsilon = u_\varepsilon & \text{on} \quad \Gamma' \cap \bar{D}_u^N\ , \end{cases}$$

where $\gamma = \gamma(x,t,N,\varepsilon)$ is any admissable (see [1], chapter 14) fitting factor.

The first theorem is a negative result, which shows that no such numerical method is satisfactory for our first class of problems. For a particular choice of boundary conditions (e.g., $\varphi(0,t) = t^2$), a suitable fitting factor could be theoretically found so that the resulting numerical method is ε-uniform for this particular choice of boundary conditions. However, the numerical method with this fitting factor will not be ε-uniform for a different set of boundary conditions (e.g., $\varphi(0,t) = t^3$).

THEOREM 1. *The fitted operator method* (P_1^N) *on a uniform rectangular mesh is not* ε*-uniform for the class of problems* (P_1) *for any choice of admissable fitting factor* γ.

PROOF. See SHISHKIN [3] for the original proof. A more readable version is given in [1], chapter 14. Extensions of the original result are given in SHISHKIN [5]. □

The second theorem shows that there is an easy fix for the above situation. We construct the following fitted mesh method using a standard finite difference operator with second order centred finite difference in space and implicit first order finite difference in time. We use a piecewise-uniform fitted rectangular mesh, which is the tensor product of a piecewise-uniform fitted mesh in space and a uniform mesh in time. To construct the piecewise-uniform mesh in space we divide Ω into three subintervals $\bar{\Omega} = \bar{\Omega}_l \cup \bar{\Omega}_c \cup \bar{\Omega}_r$, where

$$\Omega_l = (0,\sigma), \quad \Omega_c = (\sigma, 1-\sigma), \quad \Omega_r = (1-\sigma, 1).$$

The transition point σ is fitted to the left-hand and right-hand boundary layers by defining

$$\sigma = \min\{\frac{1}{4}, \sqrt{\varepsilon}\ln N\} \tag{1}$$

Then a uniform mesh with $\frac{N}{2}$ subintervals is placed on $\bar{\Omega}_c$, and uniform meshes with $\frac{N}{4}$ subintervals are placed on each of $\bar{\Omega}_l$ and $\bar{\Omega}_r$. The resulting mesh on $\bar{\Omega}$ is obviously piecewise-uniform, and it reduces to a uniform mesh whenever $\sigma = \frac{1}{4}$. The latter occurs if ε or N are sufficiently large. Note that the mesh $\bar{\Omega}_\varepsilon^N = \{x_i\}_0^{N_x}$ also depends on both N and ε. The resulting mesh on $\bar{D}_\varepsilon^N$ is defined by the tensor product

$$\bar{D}_\varepsilon^N = \bar{\Omega}_\varepsilon^N \times \{\frac{j}{N_t}T\}_0^{N_t}$$

The fitted mesh method is then

$$(P_2^N) \qquad \begin{cases} D_t^- U_\varepsilon = \varepsilon \delta_x^2 U_\varepsilon & \text{in} \quad D_\varepsilon^N \\ U_\varepsilon = u_\varepsilon & \text{on} \quad \Gamma' \cap \bar{D}_\varepsilon^N \end{cases}$$

The following theorem establishes the ε-uniform convergence of the solutions of (P_1^N) to the solution of (P_1).

THEOREM 2. *The fitted mesh method* (P_2^N) *on the Shishkin mesh* $\bar{D}_\varepsilon^N$ *with transition point* σ *defined in (1) is* ε*-uniform for the class of problems* (P_1)*. Furthermore assuming a sufficient amount of regularity and compatability on the data* φ *, the piecewise bilinear interpolant* $\bar{U}_\varepsilon$ *of the solution of* (P_2^N) *and the solution* u_ε *of problem* (P_1) *satisfy the* ε*-uniform global error estimate*

$$\sup_{0<\varepsilon\leq 1} ||\bar{U}_\varepsilon - u_\varepsilon||_{\bar{D}} \leq C(N_x^{-1}\ln N_x + N_t^{-1})$$

where C *is a constant independent of* N_x, N_t *and* ε*.*

PROOF. See SHISHKIN [3]. □

It is a remarkable fact that a trivial modification of the above ε-uniform method leads to another ε-uniform method, which has an essentially second order convergence rate with respect to the space discretisation. The required modification is the change of the transition point from that given in (1) to

$$\sigma = \min\{\frac{1}{4}, 2\sqrt{\varepsilon}\ln N\} \tag{2}$$

and the construction of the analogous piecewise-uniform mesh to the above using this new transition point. Note that the only difference in the definitions (1) and (2) is the factor 2. The fitted mesh method obtained by replacing the definition of σ in (1) by that in (2) is referred to as (P_3^N). That this simple modification makes a crucial difference is shown by the following remarkable theorem.

THEOREM 3. *The fitted mesh method* (P_3^N) *on the Shishkin mesh* $\bar{D}_\varepsilon^N$ *with transition point* σ *defined in (2) is* ε*-uniform for problem* (P_1)*. Furthermore assuming a sufficient amount of regularity and compatability on the data* φ *, the piecewise bilinear interpolant* $\bar{U}_\varepsilon$ *of the solution of* (P_2^N) *and the solution* u_ε *of problem* (P_1) *satisfy the* ε*-uniform global error estimate*

$$\sup_{0<\varepsilon\leq 1} ||\bar{U}_\varepsilon - u_\varepsilon||_{\bar{D}} \leq C((N_x^{-1}\ln N_x)^2 + N_t^{-1})$$

where C *is a constant interpolant of* N_x, N_t *and* ε*.*

PROOF. See SHISHKIN [3] for the original proof for a more general problem than (P_1). For a more readable proof for problem (P_1) see [2] □

We turn now to our second class of problems (P_2). For this problem we construct a numerical method using a standard finite difference operator and an arbitrary rectangular mesh as follows.
Let D^N be any rectangular mesh for the problem (P_2) and consider the numerical method

$$(P_4^N)\begin{cases} x_i D_t^- U_\varepsilon(x_i,t_j) = (\varepsilon\delta_x^2 - 1)U_\varepsilon(x_i,t_j) & \text{for all} \quad (x_i,t_j)\in D^N \\ U_\varepsilon = u_\varepsilon & \text{on} \quad \Gamma'\cap D^N \end{cases}$$

The following negative result shows that to obtain a satisfactory numerical method it does not always suffice to use a standard finite difference operator on a fitted mesh.

THEOREM 4. *The numerical method* (P_4^N) *comprising a standard finite difference operator on a rectangular mesh is not* ε*-uniform for problem* (P_2)*. In particular there is no fitted mesh method on a rectangular mesh which is* ε*-uniform for the class of problems* (P_2)*.*

PROOF. This is given in SHISHKIN [4]. A more readable proof may be found in [1], chapter 15. □

There is a way around this negative result. This is achieved by using both a fitted mesh and a fitted operator. An appropriate fitted-operator fitted-mesh method is

$$(P_5^N)\quad \begin{cases} x_i\gamma(x_i,t_j)D_t^- U_\varepsilon(x_i,t_j) = (\varepsilon\delta_x^2 - 1)U_\varepsilon(x_i,t_j) & \text{for all} \quad (x_i,t_j)\in D_\varepsilon^N \\ U_\varepsilon = u_\varepsilon & \text{on} \quad \Gamma'\cap \bar{D}_\varepsilon^N \end{cases}$$

where the fitting factor γ is given by

$$\gamma(x,t) = \frac{(\partial/\partial t)e^{-t/x}}{D_t^- e^{-t/x}}$$

and the Shishkin mesh D_ε^N is the same as that in the numerical method (P_3^N). The fact that this method is ε-uniform at the mesh points follows from the arguments given in Shishkin [4]; however, the piecewise-bilinear interpolant $\overline{U}_\varepsilon$ does not satisfy an ε-uniform error estimate. In Shishkin [4], a variant (P_6^N) of (P_5^N) is constructed using the same fitted finite difference operator and a rectangular fitted mesh, which is fitted not only to the parabolic boundary layers but also to the initial layer. The bilinear interpolants $\overline{U}_\varepsilon$ of the numerical solutions of (P_6^N) are shown to be ε-uniformly convergent at all points of the domain D.

4. CONCLUSION

Satisfactory numerical methods for solving singular perturbation problems always require Shishkin meshes and sometimes require, in addition, fitted operators. The two negative results above, in Theorems 1 and 4 are stated in a form which is close to the original. However, it is important to note that the positive results in Theorems 2 and 3 are stated and proved by Shishkin in much greater generality than is given here. The motivation of the present paper is to explain clearly, using the two simple concrete classes of problems (P_1) and (P_2), these remarkable results due to Shishkin. Our wish is that their significance becomes apparent to a wider audience than has heretofore been the case.

References

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